Journal of Statistical Physics, Vol. 58, Nos. 5/6, 1990

The Trajectory Scaling Function for Period Doubling Bifurcations in Flows

Maria C. de Sousa Vieira[1] **and Gemunu H. Gunaratne**[1]

Received March 27, 1989; revision received September 21, 1989

We extend the trajectory scaling function as defined for maps to flows whose dynamics is governed by ordinary differential equations. The results are obtained for the Duffing oscillator and are expected to be the same for other dissipative flows as well.

KEY WORDS: Chaos; flow; Duffing equation; scaling function.

1. INTRODUCTION

The universality of several routes to chaos has been characterized by invariants, among them scaling indices,[1,2] dimensions[3,4] [generalized dimensions, $f(\alpha)$], and trajectory scaling functions (TSFs).[5,6] They are universal for a large class of processes and help determine the universality class to which a given system belongs. The scaling indices determine the local behavior near at most a few points on the attractor, and contain no global information. The spectrum of singularities contains global information about the averaged orbit, but no local positional and dynamical information. The TSF, on the other hand, contains all local scaling information and also describes the global structure, so that it yields the most information about experimental transitions to chaos. It is also very sensitive to the exact renormalization group, and hence can be used to determine the universality class to which a given system belongs.

The TSF is a ratio of small distances in a neighborhood measured in time. By virtue of its definition it remain unchanged under smooth coordinates changes (which locally is a combination of translation, rotation, and stretching, and hence does not affect the ratio between local distances).

[1] James Franck Institute, University of Chicago, Chicago, Illinois 60637.

0022-4715/90/0300-1245$06.00/0

For example, for maps it is known that the TSF for period doubling bifurcations is a collection of step functions changing its value when the orbit comes close to a singular point. By studying the orbit, and in particular the close returns to the (generally unique) critical point, the form of the TSF can be determined perturbatively. Using a new approach, we shall study in this paper how the TSF can be calculated for period doubling bifurcations in dynamical systems described by a set of ordinary differential equations and show how the results for maps can be extended to flows.

We consider the Duffing oscillator

$$\ddot{x} + a\dot{x} + bx + cx^3 = \gamma \cos(\omega t) \tag{1}$$

which is a driven dissipative anharmonic oscillator whose parameter space has been studied in detail. We shall keep a, b, c, and w fixed and use γ as our control parameter, which leads to period doubling bifurcations. The period of the orbits in the cascade of bifurcations is given by 2^n ($n = 0, 1, 2,...$) times the period of the driving force $\gamma \cos(\omega t)$, which is $2\pi/\omega$. The values of γ where the new orbits are born converge geometrically with the Feigenbaum ratio $\delta = 4.6692...$, since we are considering a dissipative system ($a > 0$).

The aim of the present paper is to study the scaling structure present in a complete periodic orbit of a bifurcation sequence in the Duffing equation. Just as for maps, the most complete way to characterize this structure is through the trajectory scaling function. We observe that the TSF calculated for orbits with the same stability converges for n large enough. Since the period doubling phenomenon is associated with a universal theory, the TSF will be the same for a large class of continuous dynamical systems.

The paper is organized as follows: In the next section we review the definition and basic features of the TSF in maps and introduce an equivalent definition to be applied in flows. The numerical results are shown in the third section and the last gives the conclusions. The Appendix describes the method used to calculate the stability of a periodic orbit in flows.

2. THE TRAJECTORY SCALING FUNCTION

Let us first review, from refs. 5 and 6, the basic features of the TSF for period doubling bifurcations in a one-dimensional map $x_{i+1} = f(x_i)$, where f is a single-hump nonlinear function.

Consider the periodic orbit $\{x_0^{(n)}, x_1^{(n)},..., x_{N-1}^{(n)}\}$ at the nth level of the bifurcation tree ($N = 2^n$). Beyond the bifurcation every point, $x_i^{(n)}$ splits into two points $x_i^{(n+1)}$ and $x_{i+N}^{(n+1)}$ in such a way that the orbit does not close into itself after 2^n time steps, but returns exactly after $2^{(n+1)}$. For each orbit there is one value of the control parameter for which the orbit includes the critical point (peak) of the map. At this value of the parameter the derivative of the map vanishes and the cycle is called superstable. By convention, the point x_0 is chosen to be the critical point at the superstable orbit. The quantity

$$\Delta_i^{(n)} = x_i^{(n)} - x_{i+2^{n-1}}^{(n)} \tag{2}$$

depends on the stability of the orbit and represents a measure of the bifurcation. The TSF relates the scaling between two superstable orbits and is defined by

$$\sigma(i/N) = \Delta_i^{(n)}/\Delta_i^{(n+1)} \qquad (0 < i < N) \tag{3}$$

This function converges for n large enough ($n \sim 7$). The smoothness of the transformation f means that the local scalings do not change much as the map is iterated, except at certain special points, where there is a jump in σ. At these points the orbit passes close to the maximum of the map. The jumps are completely determined by the dominant exponent of the dynamical variable in f. Consequently, the TSF is a universal function which depends only on the maximum of the map.

When the map f is a diffeomorphism, $\sigma((i+1)/N)$ and $\sigma(i/N)$ are identical to the lowest order, since

$$\begin{aligned}\sigma\left(\frac{i+1}{N}\right) &= \frac{x_{i+1}^{(n)} - x_{i+1+N/2}^{(n)}}{x_{i+1}^{(n+1)} - x_{i+1+N}^{(n+1)}} \\ &= \frac{f(x_i^{(n)}) - f(x_{i+N/2}^{(n)})}{f(x_i^{(n+1)}) - f(x_{i+N}^{(n+1)})}\end{aligned} \tag{4}$$

which after a first-order Taylor expansion of $f(x)$ about x gives

$$\sigma\left(\frac{i+1}{N}\right) \approx \sigma\left(\frac{i}{N}\right) \tag{5}$$

Therefore, we see that the TSF is invariant under smooth coordinate changes and consequently $\sigma(i/N)$ can be extended to a continuous function $\sigma(u)$, where $u \in [0, 1]$. Further, $\sigma(u + 1/2) = -\sigma(u)$.

In a crude approximation the TSF for the quadratic map is given by[6]

$$\sigma(u) \approx \begin{cases} \alpha^2 & 0 < u < 1/4 \\ \alpha & 1/4 < u < 1/2 \end{cases} \tag{6}$$

where $\alpha = 2.5029...$ is the rescaling factor of the period doubling renormalization group.

Now let us extend the definition of the TSF for a continuous-time dynamical system. The Duffing equation can be rewritten as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -ay - bx - cx^3 + \gamma \cos\theta \\ \dot{\theta} &= \omega \end{aligned} \tag{7}$$

The solution of this equation for most regions of the parameter space can be found only through numerical integration. A cascade of bifurcations can be visualized in the projections into the x–y plane. In this plane a periodic orbit at the nth level of the cascade closes after a time $T_n = 2^n T_0$, where $T_0 = 2\pi/\omega$. Denote this orbit by $O^{(n)} \equiv \{\mathbf{x}^{(n)}(t) \mid 0 \leqslant t < T_n\}$, where $\mathbf{x}(t) = (x(t), y(t))$. Beyond the next bifurcation, the orbit fails to close after a time T_n but does so after $T_{n+1} = 2T_n$. The orbit $O^{(n)}$ splits into two pieces, $O_1^{(n+1)} = \{\mathbf{x}^{(n+1)}(t) \mid 0 \leqslant t < T_n\}$ and $O_2^{(n+1)} = \{\mathbf{x}^{(n+1)}(t + T_n) \mid 0 \leqslant t < T_n\}$, which almost follow each other. The distance between the two halves $O_1^{(n+1)}$ and $O_2^{(n+1)}$ of the $2^{(n+1)}$-cycle is a measure of the bifurcation and we study its scaling properties. The calculation of the TSF relating the scaling between two orbits with the same stability (as described above for maps) represents a hard numerical task. The points $\mathbf{x}^{(n)}(t + kT_0)$ and $\mathbf{x}^{(n+1)}(t + 2kT_0)$ $(k = 0, 1,...)$ will in general not lie on the same Poincaré section P_t (parametrized by t). Hence, as the time advances, the crossings of the orbits $O^{(n)}$ and $O^{(n+1)}$ with the same P_t would have to be evaluated.

For the Duffing oscillator the periodicity of the driving force implies that the points $\{\mathbf{x}^{(n+1)}(t), \mathbf{x}^{(n+1)}(t + T_0),..., \mathbf{x}^{(n+1)}(t + T_n)\}$ lie on the same Poincaré section. In this way, we can easily calculate the distances d_t between $O_1^{(n+1)}$ and $O_2^{(n+1)}$ on successive P_t's as time advances. Instead of calculating the analogue of (2), we resort to evaluating the TSF between successive branches of a single orbit, i.e., we determine how the nearest distances of $O^{(n)}$ are related to the next nearest distances. Thus, define

$$\sigma_f'\left(\frac{t}{T_n}\right) = \frac{|\mathbf{x}^{(n)}(t) - \mathbf{x}^{(n)}(t + T_{n-2})|}{|\mathbf{x}^{(n)}(t) - \mathbf{x}^{(n)}(t + T_{n-1})|} \qquad (0 < t < T_n) \tag{8}$$

where the index f in σ_f' refers to flow. All points involved in the above equation lie on P_t, and hence σ_f' is easily determined. Note that had the force not been periodic, we would not have such a simple definition of σ_f', and hence we would have to solve the intersection of $O^{(n)}$ and $O^{(n+1)}$ with a series of Poincaré sections. We believe, however, that the results obtained here will generalize to such cases.

Different Poincaré sections of an orbit are related by smooth conjugacies in general and hence in these cases $\sigma(t)$ will not change for small increases within a passage, by arguments similar to those given earlier.

We will show in Section 3 that the TSF obtained through Eq. (3) is very similar to the TSF of the quadratic map, which is a representation for Poincaré maps of the Duffing equation. However, the two-level TSF as calculated in ref. 6 shows remarkable differences between the curves obtained for maps and flows. We believe that the only reason for this is that the two distances $|\mathbf{x}^{(n)}(t)-\mathbf{x}^{(n)}(t+T_{n-1})|$ and $|\mathbf{x}^{(n+1)}(t)-\mathbf{x}^{(n+1)}(t+T_n)|$ are measured on different Poincaré sections, and this would explain the presence of many cusps on the curve of the TSF for flows. The scheme we used for a single orbit defines these distances on the same Poincaré section and this is why it is closer to the theoretical TSF.

Let us first see the TSF for one cycle in one-dimensional maps. Consider the n-fold orbit of the map. Then the TSF will be given by (ref. 6)

$$\sigma'\left(\frac{i}{N}\right)=\frac{x_i^{(n)}-x_{i+N/4}^{(n)}}{x_i^{(n)}-x_{i+N/2}^{(n)}} \qquad (0<i<N) \tag{9}$$

For n large enough ($n\sim 7$) this function converges when the orbits have the same stability. Using a similar argument as the one for σ, we can show that σ' is invariant under smooth coordinate changes; consequently, we can extend it to continuous variables by defining a new function in such way that $\sigma'(i/N)\to\sigma'(u)$. By Eq. (9) we see that it is possible to construct the complete orbit if σ' and the $N/2$ points are known. In this sense the TSF reconstruct the attractor. In order to compare with the results obtained for flows, we show $|\sigma'|$ in Fig. 1a for the quadratic map $f(x)=1-ax^2$ at the 2^{10}-superstable cycle. Observe that at every rational $u=j/2^k$, for j odd, there is a jump in $|\sigma'|$, which decreases rapidly with increasing k. The size of the discontinuities depends on the stability of the orbit. For the superstable cycle the averaged value of $|\sigma'|$ for each quartile is given by

$$|\sigma'(u)|\approx\begin{cases}9.85, & 0<u<1/4\\ 3.69, & 1/4<u<1/2\\ 6.44, & 1/2<u<3/4\\ 3.11, & 3/4<u<1\end{cases} \tag{10}$$

The ratio of the first and second quartiles of $|\sigma'(u)|$ is approximately 2.67. This number represents a estimate for the rescaling factor α.

The main advantage of this method lies in the inability to evaluate the two-level TSF from experimental data, since the stability of an experimen-

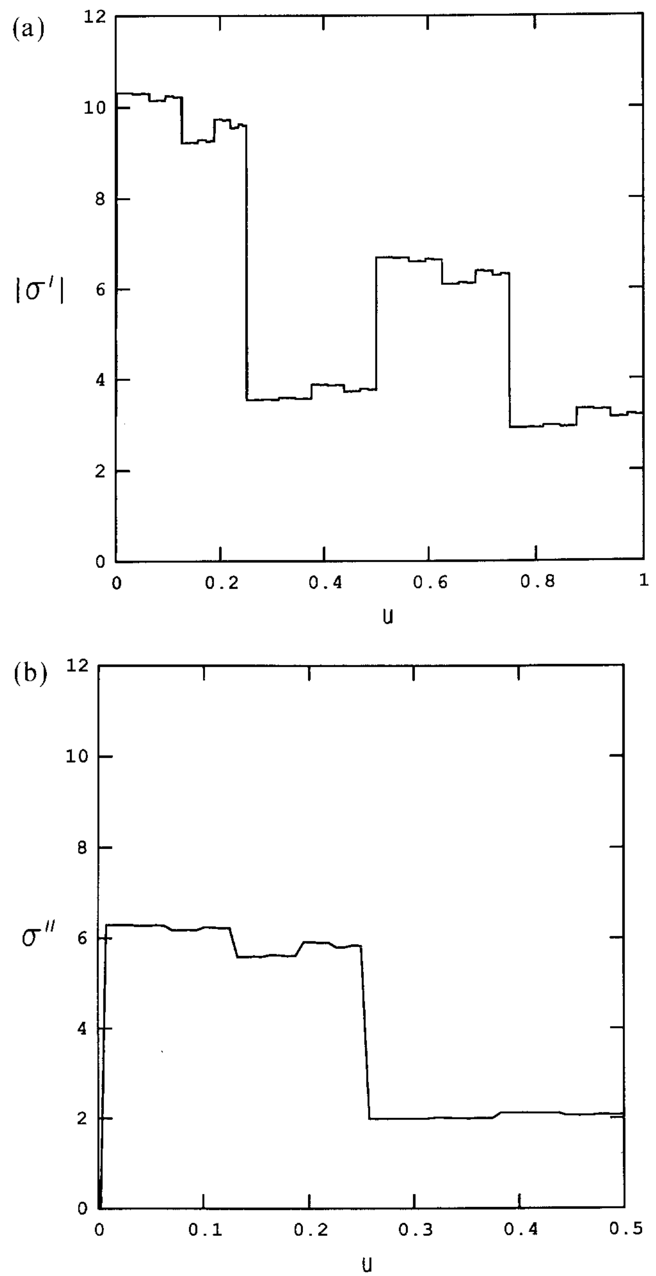

Fig. 1. The TSF for the quadratic map at the superstable cycle obtained by (a) $|\sigma'|$ with $n = 10$ and (b) σ'' with $n = 10$ and $m = 6$.

tal orbit cannot be predetermined, and hence it is not possible to get different orbits of identical stability. In fact, the TSF has only been determined recently for an experimental system,[7] and it was the single-level TSF as defined in Eq. (9).

Using one cycle, it is also possible to calculate in a reasonable approximation the TSF as defined by Eq. (3). Consider the n-fold orbit (n large) and the following definition with $m \ll n$:

$$\sigma''\left(\frac{i}{2^{m+1}}\right) = \frac{x_i^{(n)} - x_{i+2^m}^{(n)}}{x_i^{(n)} - x_{i+2^{m-1}}^{(n)}} \qquad (0 < i < 2^m) \tag{11}$$

Then, it is numerically observed that σ'' is a reasonable approximation for σ. In Fig. 1b we show the TSF for the quadratic map using the above definition. We know that in a crude approximation the first and second quartiles of σ are given by α^2 and α, respectively. In the same approximation the values of the first and second halves of σ differ from these values by 0.5% and 20% for $n = 10$ and $m = 6$.

In the next section we show the numerical calculations for the TSF in the Duffing equation using equivalent definitions to σ' and σ''.

3. NUMERICAL RESULTS

We studied the period doubling cascade of the Duffing oscillator [Eq. (7)] for the parameter values $a = 1$, $b = -10$, $c = 100$, and $w = 3.76$. The control parameter is the amplitude of the external driving force γ. We have used the Bulirsh–Stoer[8] method of numerical integration with double precision in the calculations that follow.

We show in Fig. 2 the periodic orbit in the x–y space for the 2^3-cycle at the value of γ where the orbit has its maximum stability.

The calculation of the TSF for a one-cycle orbit in a flow can be done using the definition given by Eq. (8). As mentioned before, the TSF depends on the stability of the orbit. The method used to determine the stability of a periodic orbit in a flow is described in the Appendix. The TSF for the most stable orbit is shown in Fig. 3a for $n = 8$. This function is characterized by regions where the TSF is well behaved (flat) and represent the parts that converge with increasing n. There are narrow regions where σ_f' has an oscillatory behavior with the presence of spikes. In the well-behaved parts there is a very good agreement between the equivalent curve (Fig. 1a) of the quadratic map. It means that the Poincaré sections of the Duffing equation for that section of the parameter space are well described by this map. We verified that in the flat regions the vectors $[\mathbf{x}(t) - \mathbf{x}(t + T_{n-2})]$ and $[\mathbf{x}(t) - \mathbf{x}(t + T_{n-1})]$ are almost collinear, and in

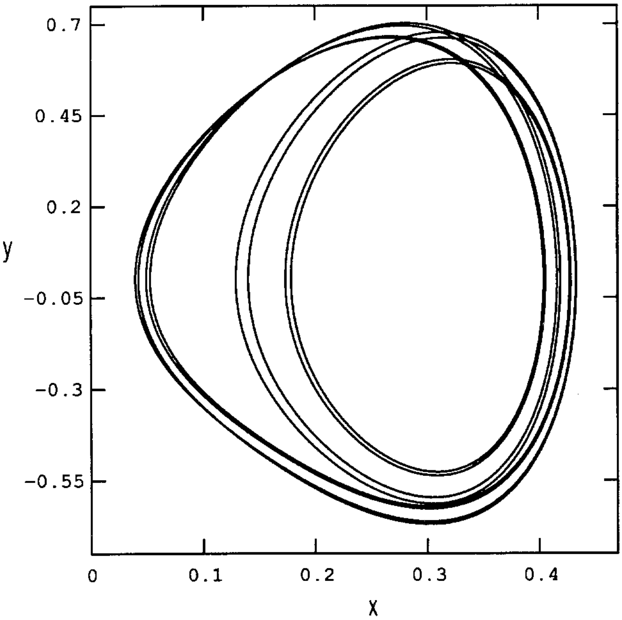

Fig. 2. The most stable 2^3-cycle for the Duffing oscillator in the x–y space for $a=1$, $b=-10$, $c=100$, and $w=3.76$.

the parts with spikes the angle ϕ between these two vectors is large. Figure 3b depicts the function $\sin^{-1}\phi$ for the same orbit.

We have also analyzed these oscillations in terms of stability. The stability of a periodic cycle can be characterized by its trace, as defined in the Appendix. We have plotted in Fig. 3c the trace of the orbit calculated after a $2\pi/\omega$ time interval. Although this calculation has not been done with the extrapolation $\Delta t \to 0$ (see the Appendix), this figure shows a qualitative picture of the stability of the orbit along the complete cycle. We observe that the regions with spikes, which correspond to large values of ϕ, are characterized by relative large fluctuations of the trace. Therefore, in these regions the stability of the orbit is very different from the global stability of the complete cycle.

The last part of our work is the calculation of the TSF using an equivalent definition to Eq. (11). If we consider the expression

$$\sigma_f''\left(\frac{t}{2T_m}\right)=\frac{|\mathbf{x}^{(n)}(t)-\mathbf{x}^{(n)}(t+T_m)|}{|\mathbf{x}^{(n)}(t)-\mathbf{x}^{(n)}(t+T_{m-1})|}\qquad (0<t<T_m)\tag{12}$$

for $m \ll n$, we obtain in a reasonable approximation the TSF for flows if it were calculated using a similar definition to Eq. (3). Figure 4 shows σ_f''

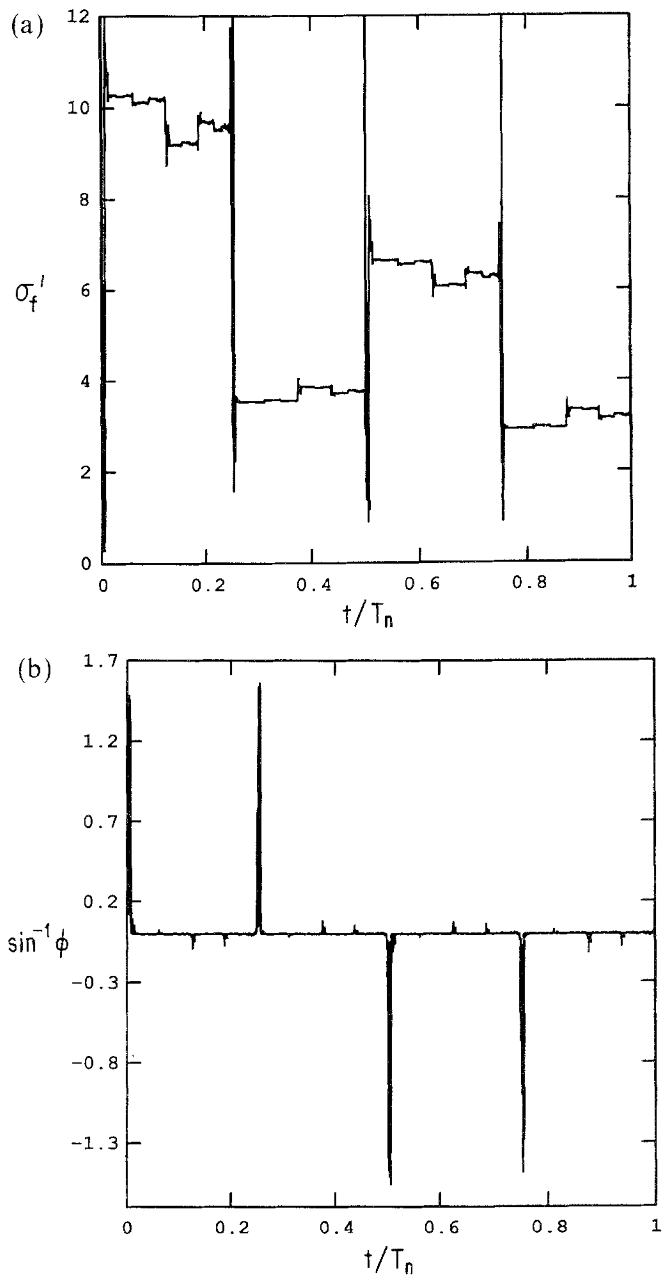

Fig. 3. (a) The TSF σ_f' for the Duffing equation calculated at the point of maximum stability of the 2^8-cycle. (b) The function $\sin^{-1}\phi$ and (c) the trace calculated at every $2\pi/\omega$ time inteval for $\Delta t = 0.001$. We have used $\gamma = 0.938541656$ and calculated 20,000 points on the x axis for (a) and (b).

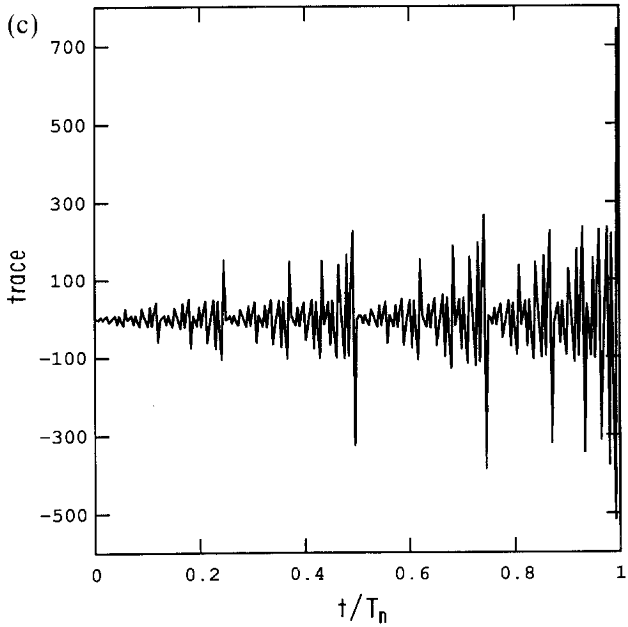

Fig. 3. (*Continued*)

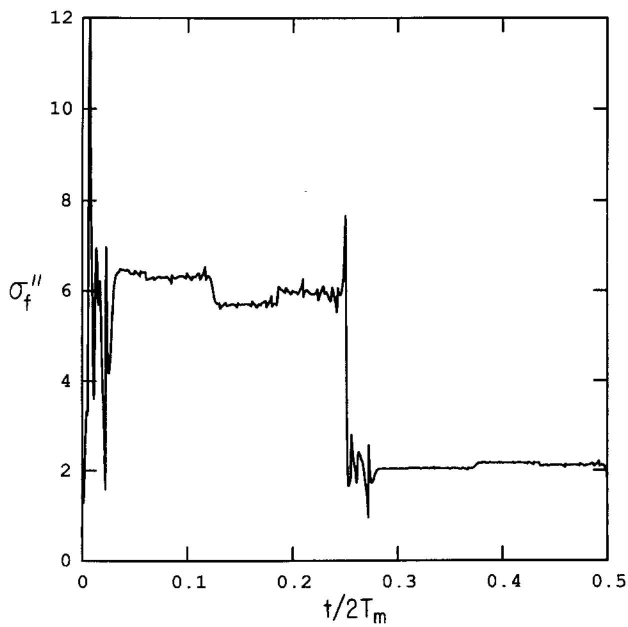

Fig. 4. The TSF σ_f'' with $m = 5$ for the Duffing equation at the most stable orbit of the 2^8-cycle.

calculated for the most stable orbit of the 2^8-cycle. The agreement with the equivalent curve for maps is good. The presence of spikes is once more observed in the regions of big discontinuities of the TSF.

4. CONCLUSIONS

We have studied the scaling structure of the period doubling bifurcations in the Duffing equation. We calculated the trajectory scaling function for one cycle, instead of using two adjacent orbits, since this represents a hard numerical task. The agreement between the TSF for the Duffing equation and the equivalent curve for the quadratic map is very good. The largest differences appear in the regions of the big jumps of the TSF. Such regions are characterized by very large fluctuations in the stability of the orbit, and by large values of the angle between the vectors $[\mathbf{x}(t)-\mathbf{x}(t+T_{n-2})]$ and $[\mathbf{x}(t)-\mathbf{x}(t+T_{n-1})]$ which are present in the definition of the TSF. To complete the work, we have used another definition of the TSF for one cycle to find an approximation for the scaling structure if it were calculated using two adjacent orbits.

APPENDIX

To determine the stability of a periodic orbit in a flow we have used the following method. Consider a set of first-order differential equations

$$\dot{\mathbf{z}}(t)=\mathbf{F}(z(t)) \tag{A.1}$$

Define $\mathbf{z}_i\equiv\mathbf{z}(t)$ and $\mathbf{z}_{i+1}\equiv\mathbf{z}(t+\Delta t)$. To first order in Δt

$$\mathbf{z}_{i+1}=\mathbf{z}_i+\mathbf{F}(\mathbf{z}_i)\,\Delta t \tag{A.2}$$

The Jacobian matrix $A^{(i)}$ of this map at the point $\mathbf{z}_i$ is $1+J(\mathbf{z}_i)\,\Delta t$, where $J(\mathbf{z}_i)$ is the Jacobian matrix of $\mathbf{F}$. For the Duffing equation (7), $A^{(i)}$ is given by

$$A^{(i)}=\begin{pmatrix} 1 & \Delta t & 0 \\ -(b+3cx_i^2)\,\Delta t & 1-a\,\Delta t & -\gamma\sin(\theta_i)\,\Delta t \\ 0 & 0 & 1 \end{pmatrix} \tag{A.3}$$

The product of the matrix $A^{(i)}$ calculated over the complete orbit gives a matrix A. We observe numerically that the elements of A have a linear correction in Δt, when they are not constants. In this way we can find by a linear extrapolation the values of these elements in the limit $\Delta t\to 0$. The eigenvalues of the extrapolated matrix provide the information about the stability of the orbit. Observe that one of the eigenvalues is one; it means that the corresponding eigenvector is tangent to the orbit. The orbit has the maximum stability when the real parts of the two other eigenvalues are zero, i.e., when the trace of the extrapolated matrix is one. We have used this method to calculate the most stable orbit for a sequence of period doubling bifurcations in the Duffing equation. The ratio between the values of γ of these orbits converges in very good agreement with the Feigenbaum constant δ.

ACKNOWLEDGMENTS

We thank Leo Kadanoff for many useful discussions. M.C.S.V. acknowledges support from the Brazilian agency CAPES, and G.H.G. from the Office of Naval Research through grant N00014-84-K-0312.

REFERENCES

1. M. J. Feigenbaum, *J. Stat. Phys.* **19**:25 (1978); **21**:669 (1979).
2. M. J. Feigenbaum, L. P. Kadanoff, and S. J. Shenker, *Physica* **5D**:370 (1982); D. A. Rand, S. Ostlund, J. Sethna, and E. D. Siggia, *Physica* **8D**:303 (1983).
3. H. G. E. Hentschel and I. Procaccia, *Physica* **8D**:435 (1983); P. Grassberger, *Phys. Lett.* **97A**:227 (1983).
4. T. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Schraiman, *Phys. Rev. A* **33**:1141 (1986).
5. M. J. Feigenbaum, *Physica* **7D**:16 (1983).
6. M. J. Feigenbaum, *Commun. Math. Phys.* **77**:65 (1980).
7. A. L. Belmont, M. J. Vinson, J. A. Glazier, G. H. Gunaratne, and B. G. Kenny, *Phys. Rev. Lett.* **61**:539 (1988).
8. J. Stoer and R. Bulirsh, *Introduction to Numerical Analysis* (Springer, New York, 1980).

Communicated by C. Tsallis